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Some notes on the study of fractals in fracture

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Abstract: In this paper, some exciting advances in the application of fractals in fracture over the last two decades are briefly reviewed and a few controversial issues are discussed. The main topics include fractality of fracture surfaces, relationships between fractal dimension and toughness, scaling and universality in fracture, fractal fracture mechanisms, and their implications for fabrication of novel materials. Also highlighted are several open problems and continuing challenges in this multidisciplinary field.

Keywords: fractal dimension, fracture toughness, scaling, size effect, universality.

1 Introduction

Fractals have been widely applied to describe the great variety of irregular, rough and fragmented natural structures that bear a special *scaling* relationship to one another [1]. It is of interest to note that the word *fractal* that was invented by Mandelbrot has the same Latin root as fracture. In their pioneering paper published in 1984, Mandelbrot *et al*. [2] showed that fracture surfaces of metals are fractal and their fractal dimensions are a measure of toughness although this is now known to be an oversimplification. Since then, experimental and simulation results have indicated that the fractality of fracture surfaces of materials, either natural or artificial, is ubiquitous [3-7]. The study on fractal fracture has been attracting much interest from scientists in materials science and solid mechanics; and as shown in Figure 1, some 50 – 60 papers have been published each year over the last decade.

As is well known, there are two challenging problems in the micromechanics of fracture: one is related to the mathematical description of microstructures, defects and damage with complex shape, size, orientation and distribution; the other is how to combine the evolution of microstructures and/or micro-damage with macro-mechanical properties [4,8]. Here, the discovery of fractal character of fracture surfaces provides both a novel method for fractography and a useful theory for building a linkage between the micro- and macro-mechanics. That is, introducing the concept of *fractal* can provide insight into understanding many important aspects in a fracture process, such as fracture precursor, the degree of predictability, and the influence of disorder at micro- and nano-scales on macroscopic properties.

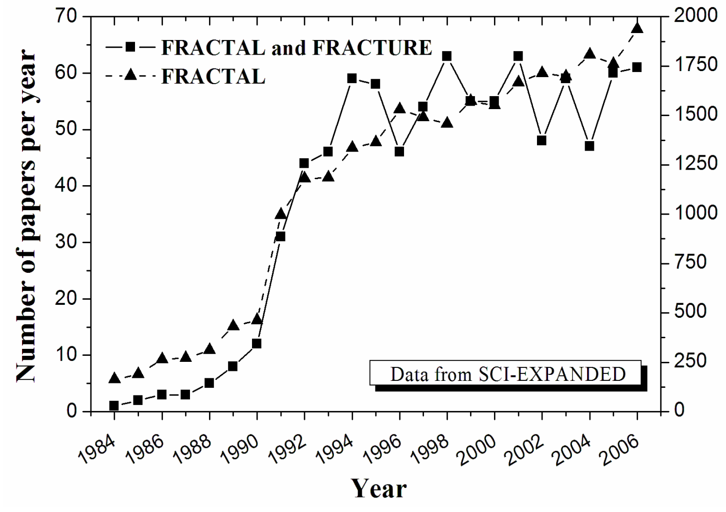

Figure 1. The number of papers published per year (1984 onwards), where the papers refer to those containing terms “Fractal” (on the right-hand axis) or “Fractal and Fracture” (on the left-hand axis) in their titles, abstracts, or keyword lists.

2 Fractals in fracture: retrospect

As with other physical phenomena, fractal structures in fracture surfaces or cracking patterns are different from those in mathematics; that is, they only apply in a certain scaling range, the upper and lower limit sizes [7]. Strictly speaking, such structures should be called as pseudo-fractals or pre-fractals. This common feature or limitation must be taken into account in the application of fractals in fracture.

2.1 Fractal dimension and toughness

The study on the morphology of fracture surfaces helps us elucidate the mechanisms of fracture, at least in a qualitative sense. One of the most widely used parameters in fractography is roughness which is defined as the ratio of actual area (or length) to its projected area (or length). Other possible definitions include the root mean square of height of asperities, the power spectrum of fracture profiles etc. Unfortunately, roughness is not a well-defined parameter since the actual area of a fracture surface is dependent on the resolution of measurement. Exploiting the concept of fractals solves this paradox on the quantitative description of roughness, and then the complexity of a fracture surface can be well measured by its fractal dimension. In general, there are two kinds of methods for measurement of fractal dimensions: changing coarse-grained levels (such as box-counting method) and using fractal measure relations such as $L^{1/D} \sim A^{1/2}$ in slit-island analysis, where D is fractal dimension, L and A are perimeter and area of islands on a polished metal fracture surface plated with nickel [2-7]. The generic property of these methods is to measure the fractal dimension of a fracture surface indirectly by decreasing one dimension although there are some methods that can be used to measure its fractal dimension directly (e.g. surface adsorption method) [5-7].

Intuitively, the rougher the fracture surface of a material, the tougher is the material. However, the intriguing fact is that, as shown in Figure 2, the relationships between roughness of fracture surfaces (measured by fractal dimensions) and toughness (such as impact energy, tearing energy, etc.) are diametrically opposite for ductile (maraging steel, titanium alloy) and brittle materials (rock, chert, and polycrystalline ceramics) [2-7,9,10]. A qualitative model was proposed in terms of the concept of *multifractal* [11]. More or less, there is a similar process in ductile and brittle fractures that can be described by the progressive creation, growth and merging of minute voids and micro-cracks respectively, but a satisfactory explanation on the piquant distinction between ductile and brittle fractures is still unavailable. Here, it is worth noting that the use of slit island analysis, a controversial but widely used method, may cause doubt about the soundness of the results in Figure 2 [12]. According to the scaling law in slit island analysis, we have the ratio, $a(d) = [L(d)]^{1/D} / [A(d)]^{1/2}$, where d is yardstick and fractal dimension $2 \le D < 3$. Thus, the yardstick d should remain a constant in measurement, but its correct choice, a hot topic in the earlier study of fractal fracture, is still an unsolved problem [6].

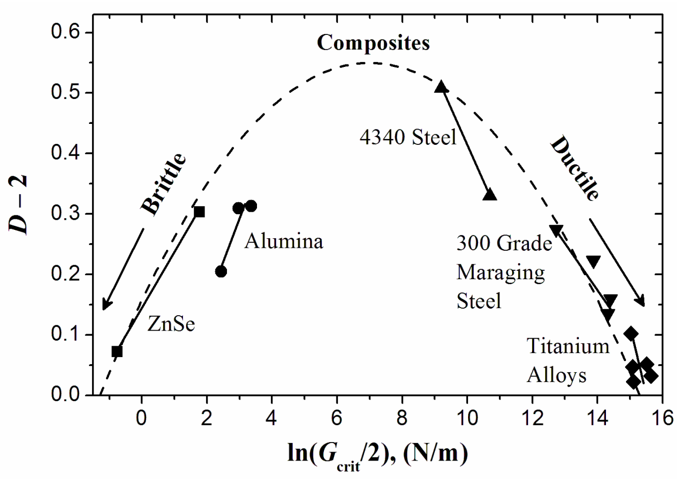

Figure 2. Experimental results of fractal dimension (D – 2) *versus* fracture energy g = $G_{crit}/2$. Note that fractal dimensions and fracture energy were obtained by using different methods under various experimental conditions (after Williford [11]).

According to Griffith and Orowan fracture theories, the critical strain energy release rate or the crack extension force is expressed as, $G_{crit} = 2\gamma_s$ for brittle fracture, and $G_{crit} = 2(\gamma_s + \gamma_p)$ for quasi-brittle fracture, where γ_s is the specific surface energy and γ_p is the plastic deformation energy associated with crack propagation at the crack tip, respectively. Considering crack propagation along zigzag (or fractal) grain boundaries such as in the intergranular fracture, the critical crack extension force can be rewritten as, $G_{crit} = 2\gamma_s \varepsilon^{2-D}$ for brittle fracture, and $G_{crit} = 2(\gamma_s \varepsilon^{2-D} + \gamma_p)$ for ductile fracture, where $\varepsilon < 1$ is a dimensionless parameter [5,6]. Alternatively, the modification for ductile fracture can be also expressed as, $G_{crit} = 2\Gamma \varepsilon^{2-D}$, where $\Gamma = \gamma_s + \gamma_p$ is the effective surface energy. The conclusion from the latter is different, or even opposite to the former due to $\gamma_p >> \gamma_s$ in most ductile fracture. This may provide a rough explanation on the negative correlation of fractal dimension of fracture surfaces *versus* toughness of ductile materials. However, it seems no doubt that, for brittle materials such as ceramics, the rougher the fracture surface (or the higher the fractal dimension), the tougher are the ceramics [10]. A direct application of fractals occurs in the design of toughened ceramics. These ceramics were designed to have a high tolerance to flaws. By adding secondary constituents such as strong, thin fibres into ceramic matrices, cracks are forced to propagate along tortuous, convoluted routes with fractal character, causing more energy to be expended than if the routes were smooth and regular [13].

2.2 Roughness exponents and universality

It was in 1990 that, in contrast to the study on relationships between fractal dimension and toughness, Bouchaud *et al.* [14] proposed that the roughness index ζ ($0 \le \zeta \le 1$, and $\zeta = 3 - D$) or fractal dimension of fracture surfaces be a universal exponent within experimental errors. The conjecture was confirmed (or disproved) by following experiments with various materials, brittle or tough, and simulations [15-17]. More interestingly, there seem two roughness exponents: $\zeta \approx 0.78$ at large length scales and $\zeta \approx 0.5$ at smaller ones, as illustrated in Figure 3. The existence of two regimes resembles to the so-called depinning transition that can be used for describing the propagation of crack fronts in heterogeneous materials [16]. The crossover length ξ separating the two regimes is dependent upon the stress intensity factor and crack velocity.

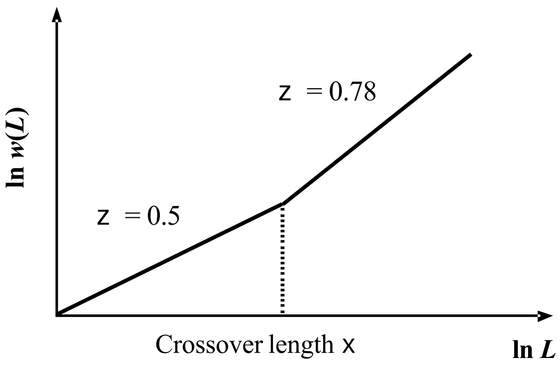

Figure 3. Log-log plot of fracture profile roughness $w(L)$ *versus* window size L, where $W(L) \sim L^{\zeta}$.

To elucidate implications of the two universal exponents and reasons on the transition from one scaling law to the other, a lot of numerical models were suggested. The simulation results have shown that, at short length scales, a quasistatic process or the minimal energy principle dominates, while at long length scales, a dynamic process or the minimal path/surface principle is of primary importance [17]. It is the competition between these two mechanisms that governs the crossover length and the roughness of fracture surfaces.

2.3 Physical mechanisms of fractal fracture

Crack propagation in a material is often considered as a typical nonlinear process occurring under nonequilibrium conditions [3,18]. In particular, similar fracture patterns formed in materials with various length scales suggest that simple network (such as spring, bond, and beam) models may provide useful results [19]. In general, the propagation of a crack is controlled by a displacement field $\boldsymbol{u}$ that follows the Navier equation: $(\lambda + \mu)\nabla[\nabla\cdot\boldsymbol{u}] + \mu\nabla^2\boldsymbol{u} = 0$, where λ and μ are the Lamé coefficients, or more simply, by the Laplace equation: $\nabla^2\phi = 0$, where ϕ is a scalar field [3,7]. The bond-breaking probability in a discretized lattice can be determined by the Metropolis or similar algorithms. Fractal cracking patterns may be generated by these models, as illustrated in Figure 4. In other words, solving partial differential equations of fracture with movable singularities or boundaries can yield fractals [1]. This is similar to patterns found in the diffusion-limited aggregation model, where random fluctuations play an important ingredient in a particle aggregation process. Unfortunately, very large amount of computer time is usually needed to obtain a cracking pattern containing just a few thousand broken bonds [19].

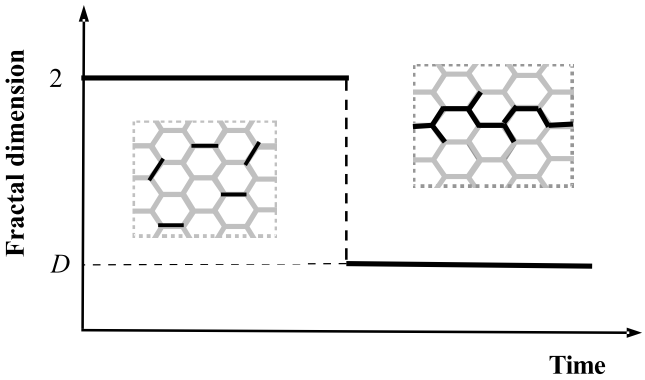

Figure 4. Illustration of a fractal fracture process, where the decrease of dimensions corresponds to the localization of damage and the final fracture [20].

Most failures in inhomogeneous materials are the result of a sequential process of nucleation, growth, and coalescence of numerous cracks or voids at micro- or nano-scales. Based on experimental findings, we developed an evolution-induced catastrophe model [21,22]. In the model, nucleated microcracks are randomly distributed in a two-dimensional lattice using Monte Carlo simulations, and the coalescence (and/or growth) condition can be approximately determined by mechanical analysis. The results showed that fracture profiles exhibit self-affine fractal characteristic with a universal roughness exponent, but, in contrast to classic percolation, the critical damage threshold is sensitive to the details of the model and the position of a crack triggering the catastrophic failure cannot be predicted *a priori* (see Figure 5). Here, the microcrack system naturally evolves into the vicinity of a critical failure due to the nonlinearly collective interaction between cracks [22]. Introducing plasticity, the macroscopic system may evolve into a critical steady state that can be described by the self-organized criticality, a theory for identifying the origin of fractal objects [23,24].

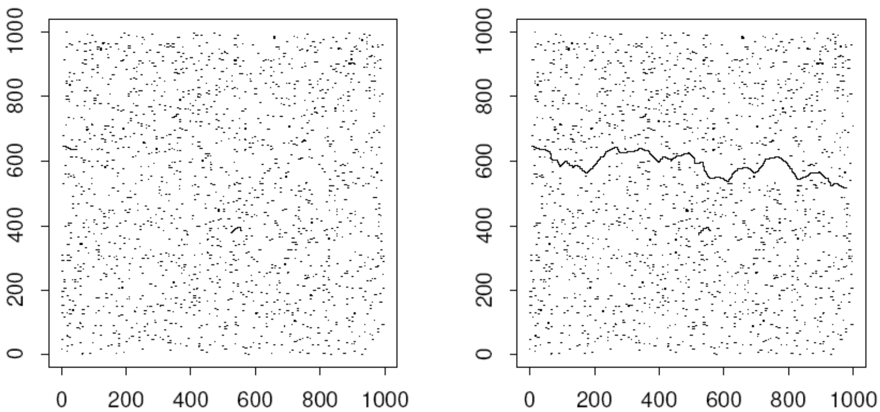

Figure 5. Patterns of microcrack evolution in a two-dimensional lattice, where a newly nucleated microcrack (*left*) triggers the catastrophic failure (*right*) [22].

3 Fractals in fracture: state-of-the-art and perspective

In recent years, there is an opinion that the study on fractal fracture is unable to answer key problems associated with fracture itself although it is still an attractive topic. In fact, this is just an illusion; the concept of fractals is still a powerful tool for both unsolved problems such as scaling laws on material strength and emerging questions in nano- and biomaterials.

3.1 Scaling and size effect

Scaling laws on the strength of solids have been one of the largest controversial issues in solid mechanics. There are two different arguments: one is based on mechanics and the other on (fractal) geometry [25-27]. The effect of specimen size on the scatter in strength measurements can be described by the Weibull statistics. In general, the scaling law on strength can be written as, $s_c \sim R^{(D-3)/2}$, where R is the characteristic length of a specimen or its defects. However, recent experiments showed that it is not always so. Especially, as the size of specimens decreases to micro- or nano-scales, there was no obvious size effect as expected or more complex strength scaling relationships appeared such as in nanoindentation [26,28]. In fine-grained metals, the critical crack extension force would rise more rapidly than that expected by Hall-Petch's $d^{-1/2}$ (d is the grain size) law. Several mechanics models such as the cohesive zone and strain-gradient plasticity have been proposed to describe these anomalies, but fractal geometry could hold a fundamental role [26].

3.2 Multiscale modelling

In contrast to quantum mechanics and relativity, in systems with an intermediate-scale (both length and time) such as fracture of inhomogeneous materials, we often miss important, sometimes even key, properties if we treat any one scale as a characteristic length and neglect others. Thus, fracture is a typical of multiscale problems where parameters with the same dimension but of largely different magnitude enter the models of the phenomenon simultaneously [4,29]. Here, a prevailing method is to incorporate the smaller-scale physics into continuum mechanics models. However, as the scale enters micro- and nano-levels, more details in both mechanical properties and geometrical structures need to be considered. So three, four and even more model constants appear. This will increase the complexity of the models, and often lose a clear physical sense for these parameters. Hence, some unique methods in applies mathematics such as intermediate asymptotics, complete and incomplete similarity (or fractal) and in modern physics such as percolation and renormalization group, could provide a powerful tool for the analysis of multiscale phenomena [4,30,31].

3.3 Fractal and nanomechanics

Compared with their traditional counterparts, nanostructured materials exhibit superior physical and mechanical properties with far less filler contents. It is generally believed that the improvement of their properties is mainly caused by very large surface areas between fillers and matrices. For example, breaking the whole clay particulate into nano-platelets, the increase of surface areas is proportional to aspect ratios (10 – 1000) of exfoliated platelets. In fact, these unique nanostructures are similar to those so-called "monster" structures in fractal geometry [1,3]. Recent studies showed that fractal and relevant concepts can be well applied to describe the anomalous properties in nanomaterials such as barrier in polymer-clay nanocomposites and superhardness in nanocomposite ceramic coatings [32,33].

Biological materials with nanostructures are the product of nature formed from billions of years of natural evolution and competition for survival. Also, fractal, the geometry of nature, has been used to gain a better understanding of their novel mechanical properties and physical mechanisms, which has been a hot topic in nanomechanics. Using a fractal bone model, one could answer questions about why the fundamental building blocks of biological materials are generally designed at the nanoscale and why nanoscale is so important to biological materials [34,35].

4 Conclusions

In this brief review, we have touched on some of the advances made in the application of fractals in fracture over the past two decades, showing that fractal geometry has played an important role in this multidisciplinary field. Although significant progress has been achieved, the study in this field is still in its infancy and many aspects associated with fractal fracture itself are still unclear. In particular, the study on disorder and fracture may become a new joint research area spanning materials physics and solid mechanics. At least, the following issue is worthy of further systematic investigation: can we hope to predict, long in advance, mechanical failure of complex structural materials? Or, conversely, might we discover that the complexity of nanostructures imposes intrinsic limits to our ability to predict their behaviour? It is hoped that in the foreseeable future fractal concepts may help shed light on these open and challenging problems.

Acknowledgements

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